

Ex 1 Let $x(n) = a^n u(n)$, $\begin{cases} |a| < 1 \\ a \text{ is real} \end{cases}$, Find Spectrum of $x(n)$.

Sol

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n}$$

$\downarrow$
 $n \geq 0$

Thus

$$X(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

$$X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}, \quad |X(\Omega)| = \frac{1}{\sqrt{1 - 2a \cos(\Omega) + a^2}}$$

We can also write

$$X(\Omega) = \frac{1 - a e^{+j\Omega}}{(1 - a e^{-j\Omega})(1 - a e^{+j\Omega})} = \frac{1 - a e^{j\Omega}}{1 - 2a \cos(\Omega) + a^2}$$

$$\angle X(\Omega) = \varphi(\Omega) = \tan^{-1} \left(\frac{-a \sin(\Omega)}{1 - a \cos(\Omega)} \right)$$

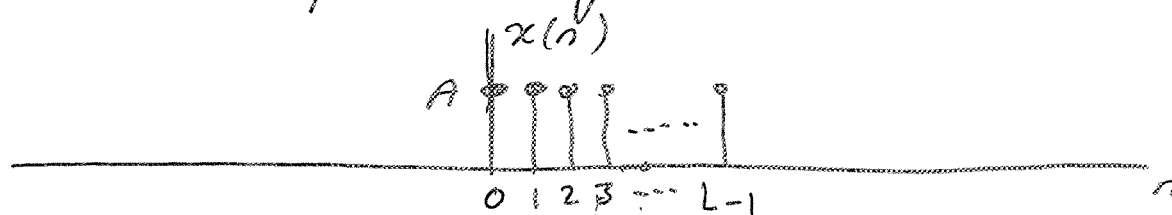
$$\text{or } \varphi(\Omega) = - \tan^{-1} \left(\frac{a \sin(\Omega)}{1 - a \cos(\Omega)} \right).$$

Note $X(\Omega)$, and thus $|X(\Omega)|$ and $\varphi(\Omega)$ are periodic in frequency Ω with fundamental period 2π . That is

$$X(\Omega) = X(\Omega \pm 2\pi).$$

Also

$$\underbrace{|X(\Omega)| = |X(-\Omega)|}_{\text{even}}, \quad \underbrace{\varphi(\Omega) = -\varphi(-\Omega)}_{\text{odd}}$$

Ex 2Find spectrum of $x(n)$ where

Sol

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = \sum_{n=0}^{L-1} A e^{-j\Omega n}$$

$$X(\Omega) = A \sum_{n=0}^{L-1} e^{-j\Omega n} = A \frac{1 - e^{-j\Omega L}}{1 - e^{-j\Omega}}$$

$$X(\Omega) = A \left(\frac{e^{-j\Omega \frac{L}{2}}}{e^{-j\Omega \frac{1}{2}}} \right) \frac{e^{+j\Omega \frac{L}{2}} - e^{-j\Omega \frac{L}{2}}}{e^{+j\Omega \frac{1}{2}} - e^{-j\Omega \frac{1}{2}}}$$

$$X(\Omega) = A \left(e^{-j\Omega \left(\frac{L-1}{2} \right)} \right) \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

$$|X(\Omega)| = |A| \cdot \frac{\left| \sin\left(\frac{\Omega L}{2}\right) \right|}{\left| \sin\left(\frac{\Omega}{2}\right) \right|}$$

$$\varphi(\Omega) = -\Omega \left(\frac{L-1}{2} \right) + \begin{cases} 0 & \text{if } \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} > 0 \\ \pm\pi & \text{if } \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} < 0 \\ \text{undefined} & \text{if } \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} = 0 \end{cases}$$

for any Ω .

Properties of Discrete-Time Fourier Transform (DTFT)

(1) Periodicity:

$$X(\Omega \pm 2\pi) = X(\Omega), \quad 2\pi \text{ is fundamental period in frequency domain.}$$

(2) Linearity:

$$\begin{aligned} \text{If } x_1(n) &\xleftrightarrow{\text{DTFT}} X_1(\Omega) \\ \text{and } x_2(n) &\xleftrightarrow{\text{DTFT}} X_2(\Omega) \end{aligned}$$

$$\text{then } v(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{\text{DTFT}} V(\Omega) = a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

(3) Time shifting:

$$\text{If } x(n) \longleftrightarrow X(\Omega)$$

$$\text{then } v(n) = x(n \pm k) \longleftrightarrow e^{\pm j\Omega k} X(\Omega) = V(\Omega)$$

(4) Frequency shifting (or Complex Modulation)

$$\text{If } x(n) \longleftrightarrow X(\Omega) \text{ then}$$

$$v(n) = e^{\pm j\Omega_0 n} x(n) \longleftrightarrow V(\Omega) = X(\Omega \mp \Omega_0)$$

(5) Real Modulation:

If $x(n) \longleftrightarrow X(\Omega)$ then

$$v(n) = \cos(\Omega_0 n) x(n) \longleftrightarrow \frac{1}{2} X(\Omega - \Omega_0) + \frac{1}{2} X(\Omega + \Omega_0)$$

Because $\cos(\Omega_0 n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n}$ and using property (4).

(6) Differentiation in Frequency:

If $x(n) \longleftrightarrow X(\Omega)$ then

$$v(n) = n x(n) \longleftrightarrow V(\Omega) = j \frac{dX(\Omega)}{d\Omega}$$

(7) Convolution in time domain:

If $x(n) \longleftrightarrow X(\Omega)$ and $h(n) \longleftrightarrow H(\Omega)$
then

$$y(n) = x(n) * h(n) \longleftrightarrow Y(\Omega) = X(\Omega) H(\Omega)$$

Ex 1 Find spectrum of $y(n) = n a^{+n} u(n)$, $\begin{cases} |a| < 1 \\ a \text{ real} \end{cases}$

Sol

we let $x(n) = a^{+n} u(n)$ then $X(\Omega) = \frac{1}{1 - a e^{j\Omega}}$ as obtained before (in page 7-11). then

$$y(n) = n x(n) \longleftrightarrow Y(\Omega) = j \frac{dX(\Omega)}{d\Omega}$$

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

$$\frac{dX(\Omega)}{d\Omega} = \frac{j e^{j\Omega} (e^{j\Omega} - a) - j e^{j\Omega} e^{j\Omega}}{(e^{j\Omega} - a)^2} = \frac{-a j e^{j\Omega}}{(e^{j\Omega} - a)^2}$$

and

$$Y(\Omega) = j \frac{dX(\Omega)}{d\Omega} = \frac{a e^{j\Omega}}{(e^{j\Omega} - a)^2} = \frac{a e^{-j\Omega}}{(1 - a e^{-j\Omega})^2}$$

Ex 2 Find $y(n) = x(n) * h(n)$ where

$$h(n) = a^{+n} u(n), -1 < a < 1 \text{ and } x(n) = \{1, 0, -2, 3\}$$

Sol

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = 1 - 2 e^{-j2\Omega} + 3 e^{-j3\Omega}$$

$$H(\Omega) = \frac{1}{1 - a e^{-j\Omega}}$$

and

$$Y(\Omega) = H(\Omega) \cdot X(\Omega) = \frac{1}{1 - a e^{-j\Omega}} - 2 e^{-j2\Omega} \frac{1}{1 - a e^{-j\Omega}} + 3 e^{-j3\Omega} \frac{1}{1 - a e^{-j\Omega}}$$

Thus knowing inverse of $X(\Omega)$ and property (3) we have

$$y(n) = \underbrace{a^{+n} u(n)}_{x(n)} - 2 \underbrace{a^{+(n-2)} u(n-2)}_{x(n-2)} + 3 \underbrace{a^{+(n-3)} u(n-3)}_{x(n-3)}$$

(8) Modulation:

If $x_1(n) \longleftrightarrow X_1(\Omega)$ and $x_2(n) \longleftrightarrow X_2(\Omega)$

then

$$v(n) = x_1(n) x_2(n) \longleftrightarrow V(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta) X_2(\Omega - \theta) d\theta$$

(9) If $x(n)$ is real then for

$$x(n) \longleftrightarrow X(\Omega)$$

we have $|X(\Omega)| = |X(-\Omega)|$

$$\angle \phi(\Omega) = -\angle \phi(-\Omega)$$

(10) If $x(n)$ is real and even then $X(\Omega)$ is real

too, And

$$\angle \phi(\Omega) = \begin{cases} 0 & \text{if } X(\Omega) > 0 \text{ for any } \Omega \\ \pm \pi & \text{if } " < 0 " " " \\ \text{undefined} & \text{if } " = 0 " " " \end{cases}$$

Energy of $x(n)$

If $x(n)$ is energy signal then

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty, \text{ i.e. } E_x \text{ has finite value.}$$

Now,

$$E_x = \sum_{n=-\infty}^{\infty} x(n) x^*(n) = \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} X(-\Omega) e^{-j\Omega n} d\Omega$$

Since $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$

Then

$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(-\Omega) \underbrace{\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \right]}_{X(\Omega)} d\Omega$$

Thus

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(-\Omega) X(\Omega) d\Omega$$

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$$

Parseval's theorem

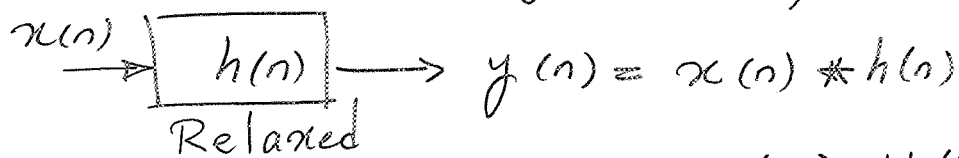
* Energy Density Function (or spectrum) of $x(n)$ is real and even function:

$$\begin{aligned} |X(\Omega)|^2 &= S_{xx}(\Omega) \quad \text{for } -\pi \leq \Omega \leq \pi \\ &= S_{xx}(-\Omega) \quad \text{i.e. in one period.} \end{aligned}$$

* Energy of $x(n)$ in Frequency band of

$\Omega_1 < \Omega < \Omega_2$ is

$$\frac{1}{2\pi} \int_{\Omega_2}^{\Omega_1} |X(\Omega)|^2 d\Omega$$

LTI system, Frequency Analysis

$$Y(\Omega) = X(\Omega) H(\Omega)$$

$$|Y(\Omega)| = |X(\Omega)| |H(\Omega)|$$

$$\angle Y(\Omega) = \angle X(\Omega) + \angle H(\Omega)$$

$$|Y(\Omega)|^2 = |X(\Omega)|^2 |H(\Omega)|^2$$

Case 1: Let $x(n) = A e^{j\Omega_0 n}$ for a given/known frequency Ω_0 and $-\infty < n < \infty$.

That is $x(n)$ is periodic.

Also let $h(n) \longleftrightarrow H(\Omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n}$

We want to find $y(n)$.

Sol $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$

by substituting for $x(n-k)$ we obtain

$$\begin{aligned}
 y(n) &= \sum_{k=-\infty}^{\infty} h(k) A e^{j\Omega_0(n-k)} \\
 &= A e^{j\Omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-j\Omega_0 k}}_{H(\Omega_0)}
 \end{aligned}$$

Thus

$$y(n) = A e^{j\Omega_0 n} H(\Omega_0) \quad \text{for } -\infty < n < \infty$$

y(n) is product of input and H(Ω₀).

y(n) is periodic like the input with the same frequency, Ω₀.

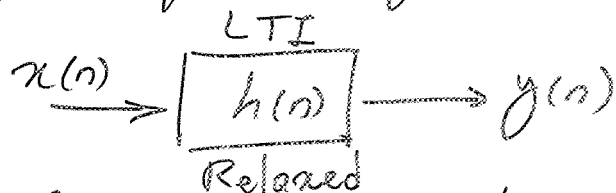
For $H(\Omega_0) = |H(\Omega_0)| e^{j\varphi(\Omega_0)}$, then

$$y(n) = A |H(\Omega_0)| e^{j(\Omega_0 n + \varphi(\Omega_0))}$$

That is the system $h(n) \longleftrightarrow H(\Omega)$ modified the amplitude of the input, and add phase to the phase of the input appearing at the output, y(n).

Case 2

Let $x(n) = A \cos(\Omega_0 n + \theta_0)$, Find output $y(n)$ of the system $h(n)$.

Sol

$$x(n) = \frac{A}{2} e^{j(\Omega_0 n + \theta_0)} + \frac{A}{2} e^{-j(\Omega_0 n + \theta_0)}$$

Using results of Case 1, we have

$$y(n) = \frac{A}{2} |H(\Omega_0)| e^{j(\Omega_0 n + \theta_0 + \phi(\Omega_0))} + \frac{A}{2} |H(-\Omega_0)| e^{-j(\Omega_0 n + \theta_0 - \phi(-\Omega_0))}$$

But $|H(\Omega_0)| = |H(-\Omega_0)|$ and $\phi(\Omega_0) = -\phi(-\Omega_0)$

Thus

$$y(n) = \frac{A}{2} |H(\Omega_0)| \left\{ e^{j(\Omega_0 n + \theta_0 + \phi(\Omega_0))} + e^{-j(\Omega_0 n + \theta_0 + \phi(\Omega_0))} \right\}$$

or

$$y(n) = A |H(\Omega_0)| \cos(\Omega_0 n + \theta_0 + \phi(\Omega_0))$$

where $h(n) \longleftrightarrow H(\Omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n} = |H(\Omega)| e^{j\phi(\Omega)}$

Result Since the system $h(n)$ is LTI then for the input

$$x(n) = \sum_{k=1}^L A_k \cos(\Omega_k n + \theta_k)$$

we get

$$y(n) = \sum_{k=1}^L A_k |H(\Omega_k)| \cos[\Omega_k n + \theta_k + \phi(\Omega_k)]$$

Ex 1 Let $h(n) = a^n u(n)$, $-1 < a < 1$ and the input be

$$x(n) = 5 - 2 \cos\left(\frac{3\pi}{4}n - \frac{\pi}{6}\right) + 3 \sin\left(\frac{\pi}{4}n\right) + e^{j\left(\frac{\pi}{3}n\right)}$$

Find the output $y(n) = x(n) * h(n)$. for $-\infty < n < \infty$

Sol

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

$$H(\Omega) = \sum_{n=0}^{\infty} (a e^{-j\Omega})^n = \frac{1}{1 - a e^{-j\Omega}}$$

Thus

$$y(n) = 5 H(\Omega=0) - 2 H(\Omega=\frac{3\pi}{4}) \cos\left(\frac{3\pi}{4}n - \frac{\pi}{6}\right) + 3 H(\Omega=\frac{\pi}{4}) \sin\left(\frac{\pi}{4}n\right) + H(\Omega=\frac{\pi}{3}) e^{j\left(\frac{\pi}{3}n\right)}$$

Now $H(\Omega=0) = \frac{1}{1-a}$

$$H(\Omega = \frac{3\pi}{4}) = \frac{1}{1 - a e^{-j\frac{3\pi}{4}}} = |H(\frac{3\pi}{4})| e^{j\varphi(\frac{3\pi}{4})}$$

$$H(\Omega = \frac{\pi}{4}) = \frac{1}{1 - a e^{-j\frac{\pi}{4}}} = |H(\frac{\pi}{4})| e^{j\varphi(\frac{\pi}{4})}$$

$$H(\Omega = \frac{\pi}{3}) = \frac{1}{1 - a e^{-j\frac{\pi}{3}}} = |H(\frac{\pi}{3})| e^{j\varphi(\frac{\pi}{3})}$$

$$|H(\Omega)| = \frac{1}{\sqrt{1 - 2a \cos(\Omega) + a^2}}, \quad \varphi(\Omega) = -\tan^{-1}\left(\frac{a \sin(\Omega)}{1 - a \cos(\Omega)}\right)$$

Ex 2

Find $y(n)$, for $-\infty < n < \infty$, of the relaxed causal system

$$y(n] = \frac{1}{2} y(n-1) + x(n), \text{ where}$$

$x(n)$ is that given in Ex 1 (page 7-22).

Sol we must find $h(n)$ from the difference equation first:

For $x(n] = \delta(n)$ and zero-initial conditions

we find the output to be

$$h(n) = \left(\frac{1}{2}\right)^n u(n).$$

Now
$$H(\Omega) = \sum_{n=0}^{\infty} h(n) e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n}$$

$$H(\Omega) = \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

Now just like Ex1 we find $y(n)$, using $H(\Omega)$.

Ex 3 Consider the relaxed causal system

$$y(n] = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n]$$

Find $y(n]$ for $-\infty < n < \infty$ where

$$x(n] = \sqrt{2} + 3 \cos\left(\frac{\pi}{4}n - \frac{\pi}{6}\right), \quad -\infty < n < \infty$$

Sol

step 1 Find $h(n]$ of the system by letting $x(n] = \delta(n]$ and assuming zero-initial conditions.

$$h(n] = \frac{5}{6} h(n-1) - \frac{1}{6} h(n-2) + \delta(n]$$

Homogeneous solution:

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0 \Rightarrow \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{3}\right) = 0$$

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{3}$$

$$h(n] = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n \quad \text{for } n \geq 0$$

Particular solution is zero.

Thus

$$\begin{cases} h(n) = c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{3}\right)^n \\ h(n) = \frac{5}{6} h(n-1) - \frac{1}{6} h(n-2) + \delta(n) \end{cases}$$

For

$$n=0 \Rightarrow \begin{cases} h(0) = c_1 + c_2 \\ h(0) = 0 + 0 + 1 = 1 \end{cases} \Rightarrow \boxed{c_1 + c_2 = 1}$$

$$n=1 \Rightarrow \begin{cases} h(1) = \frac{1}{2} c_1 + \frac{1}{3} c_2 \\ h(1) = \frac{5}{6} h(0) + 0 + 0 = \frac{5}{6} \end{cases}$$

$$\Rightarrow \boxed{\frac{1}{2} c_1 + \frac{1}{3} c_2 = \frac{5}{6}}$$

Thus

$$\begin{cases} c_1 + c_2 = 1 \\ 3c_1 + 2c_2 = 5 \end{cases} \Rightarrow \begin{cases} \boxed{c_1 = 3} \\ \boxed{c_2 = -2} \end{cases}$$

and $\boxed{h(n) = 3 \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n}$ for $n \geq 0$

$$H(\Omega) = \sum_{n=0}^{\infty} h(n) e^{-j\Omega n} = \frac{3}{1 - \frac{1}{2} e^{-j\Omega}} - \frac{2}{1 - \frac{1}{3} e^{-j\Omega}}$$

Thus

$$y(n) = \sqrt{2} H(0) + 3 H\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}n - \frac{\pi}{6}\right)$$

where $H(0) = \frac{3}{1 - \frac{1}{2}} - \frac{2}{1 - \frac{1}{3}} = 3$

$$H\left(\frac{\pi}{4}\right) = \frac{3}{1 - \frac{1}{2} e^{-j\frac{\pi}{4}}} - \frac{2}{1 - \frac{1}{3} e^{-j\frac{\pi}{4}}}$$

$$H\left(\frac{\pi}{4}\right) = \frac{3}{1 - \frac{1}{2} \left(\frac{\sqrt{2}}{2} - j \frac{\sqrt{2}}{2} \right)} - \frac{2}{1 - \frac{1}{3} \left(\frac{\sqrt{2}}{2} - j \frac{\sqrt{2}}{2} \right)}$$

$$H\left(\frac{\pi}{4}\right) = \frac{12}{4 - (\sqrt{2})(1-j)} - \frac{12}{6 - (\sqrt{2})(1-j)}$$

$$H\left(\frac{\pi}{4}\right) = \frac{24}{24 - 10\sqrt{2} + j(10\sqrt{2} - 4)}$$

--> See Tables in pages 391 and 392 of Oppenheim book.